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Connectedness of a class of planar self-affine tiles ☆

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ABSTRACT

We consider a class of planar self-affine tiles T that are generated by the lower triangular expanding matrices and the product-form digit sets. We give necessary and sufficient conditions for T to be connected and disk-like. Also for the disconnect case, we give a condition that enumerates the number of connected components of T.

1. Introduction

Let A be a $d \times d$ expanding matrix (all its eigenvalues have moduli > 1). It is well known that for any finite set $\mathcal{D} \subset \mathbb{R}^d$, there exists a unique compact set $T := T(A, \mathcal{D})$ such that

$$T = \bigcup_{d \in \mathcal{D}} A^{-1}(T + d), \tag{1.1}$$

i.e., $AT = \bigcup_{d \in \mathcal{D}}(T + d)$; also T has the expression

$$T = \left\{ \sum_{k=1}^{\infty} A^{-k} d_k \colon d_k \in \mathcal{D} \right\}. \tag{1.2}$$

We call $\mathcal{D}$ a *digit set* and the pair $(A, \mathcal{D})$ a *self-affine pair*. If $\#\mathcal{D} = |\det(A)|$ is an integer and T has non-void interior, then T actually tiles $\mathbb{R}^d$ in the following sense [3]: there exists a discrete set $\mathcal{J} \subseteq \mathbb{R}^d$ which satisfies (i) $T + \mathcal{J} = \mathbb{R}^d$ and (ii) $(T^o + u) \cap (T^o + v) = \emptyset$ for all distinct $u, v \in \mathcal{J}$. We call such T a *self-affine* tile.

The class of self-affine tiles has very rich analytic and number theoretic properties. It has attracted a lot of attention in fractal geometry, wavelet theory and number theory, and there is a wealth of literature on this topic (see e.g., [3,13–15,7,18,1,10,6]). On the other hand, the topological property such as the connectedness, the disk-likeness and the structure of the components (when disconnected) of the self-affine tiles is still not well established. Various results can be found in Bandt and Gelbrich [4], Bandt and Wang [5], Gröchenig and Haas [7], Hacon et al. [8] and Akiyama and Thuswaldner [1]. Recently Kirat and one of the authors initiated a more systematic study for the tiles that are generated by consecutive collinear digit sets [11], a direct analog of the one-dimensional case. It was found that the connectedness depends solely on

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the characteristic polynomial of the expanding matrix, and by using that one can establish the connectedness of such tiles up to dimension 4 [12,2]. The higher dimensional case is still unsolved [9]. Along this line, the disk-likeness of such tiles in $\mathbb{R}^2$ has also been characterized [16].

In an attempt to consider the tiles that are arisen from the more general digit sets, we try another class of simple cases: the matrix A is lower triangular and the digit set is arranged into a rectangular form. It is interesting to find that the resulting tiles can be disconnected even for the very simple matrices. This will be a good pilot case for the more general self-affine tiles. Among the other results, we prove

Theorem 1.1. *Let p, q be integers with $|p|, |q| \geqslant 2$ and let $a \in \mathbb{R}$. Let*

$$A = \begin{bmatrix} p & 0 \\ -a & q \end{bmatrix}, \qquad \mathcal{D} = \left\{ \begin{bmatrix} i \\ j \end{bmatrix} : 0 \leqslant i \leqslant |p|-1,\ 0 \leqslant j \leqslant |q|-1 \right\}. \tag{1.3}$$

Then T is a self-affine tile, and it is connected if and only if $|\frac{a}{q(q-\operatorname{sgn}(p))}| \leqslant 1$. (Here $\operatorname{sgn}(p)$ denotes the sign of p.)

Moreover T is disk-like if and only if the above $\leqslant$ is replaced by $<$.

If the tile T is disconnected, then we have a precise count on the connected components.

Theorem 1.2. *Let T be the self-affine tile as in Theorem 1.1. Suppose $|q|^{m-1} < |\frac{a}{q(q-\operatorname{sgn}(p))}| \leqslant |q|^m$. Then T has $|p|^m$ connected components.*

The proof of the theorems is based on the radix expansion of elements of T: the expression of (1.2) is reduced to (as a special case of (2.5))

$$T = \left\{ \begin{bmatrix} p(\mathbf{i}) \\ ar(\mathbf{i}) \end{bmatrix} + \begin{bmatrix} 0 \\ q(\mathbf{j}) \end{bmatrix} : 0 \leqslant i_n < p,\ 0 \leqslant j_n < q \right\} \tag{1.4}$$

where $p(\mathbf{i}) = \sum_n \frac{i_n}{p^n}$, $q(\mathbf{j}) = \sum_n \frac{j_n}{q^n}$, and the term $r(\mathbf{i})$ comes from the sub-diagonal entry. It follows that the range of the x-coordinate of T is the interval $[0,1]$. The tile T is bounded by two vertical segments at $x = 0$ and $x = 1$ with length 1, the upper and lower sides are serrated edges. For those x with two representations, say, $x = p(\mathbf{i}) = p(\mathbf{i}')$ with $\mathbf{i} \neq \mathbf{i}'$, the x-cross section of T is the union of two intervals of length 1 with the lower end points at $ar(\mathbf{i})$ and $ar(\mathbf{i}')$ respectively. For small $|a|$, the tile is connected since any two such intervals intersect. By increasing $|a|$, these pairs of intervals shift up or down so that they change from intersecting to non-intersecting. This makes the tile changes from connected to disconnected, and the number of connected components increases. The precise classification in the theorems is established through some careful calculation of radix expansions of $ar(\mathbf{i}) + q(\mathbf{j})$ of the y-coordinate.

For an illustration of the two theorems, we let

$$A = \begin{bmatrix} 2 & 0 \\ -a & 2 \end{bmatrix} \quad \text{and} \quad \mathcal{D} = \left\{ \begin{bmatrix} i \\ j \end{bmatrix} : i, j \in \{0,1\} \right\}.$$

The four pictures are the graphs of the tiles T for $a = 1, 2, 3, 5$ respectively (see Fig. 1), they have width 1, and the two vertical sides at the ends have length 1 (note that the vertical scales in the pictures have been adjusted). The tile is a union of four affine copies (see (1.1)), the black region is one of the affine copy $A^{-1}T$.

It follows from Theorem 1.1 that T is connected for the cases $a = 1$ and $a = 2$, and by Theorem 1.2, T has two components when $a = 3$, and four components when $a = 5$. Indeed by inspecting the graph for $a = 3$, it is clear that each of the four affine copies $A^{-1}T + d$, $d \in \mathcal{D}$ is disconnected, but the union of the two on top of each other makes up a connected component. For $a = 5$, we use a similar observation for $T = \bigcup A^{-2}(T + A\alpha + \beta)$, $\alpha, \beta \in \mathcal{D}$ (apply (1.1) one more time), each connected component is the union of four of these sub-tiles on tops of each other.

We remark that the above tile has its own interest on the periodicity of the tiling set $\mathcal{J}$. It was studied by Lagarias and Wang in [13]: let $A = \left[\begin{smallmatrix} 2 & 0 \\ -a & 2 \end{smallmatrix}\right]$ and $\mathcal{D} = \left\{ \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right] \right\}$, then all *self-replicating tilings* $\mathcal{J}$ of T are non-periodic, on the other hand T admits a non-self-replicating tiling that is a lattice ($\mathcal{J} = \mathbb{Z} \oplus 3\mathbb{Z}$).

For the organization of the paper, we prove the connectedness of the aforementioned tiles in Section 2 (Theorem 2.1) and the disk-likeness in Section 3 (Theorem 3.1). Actually we can put the digit sets in a slightly more general setting (as in (2.1)) without causing much difficulty. The proof of Theorem 1.2 is given in Section 4. In Section 5 we give some remarks concerning the more general situations.

2. The connectedness

The digit sets in Theorem 1.1 can actually be put into a more general form:

$$\mathcal{D} = \left\{ \begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} 0 \\ b_i \end{bmatrix} : 0 \leqslant i \leqslant |p|-1,\ 0 \leqslant j \leqslant |q|-1 \right\}. \tag{2.1}$$

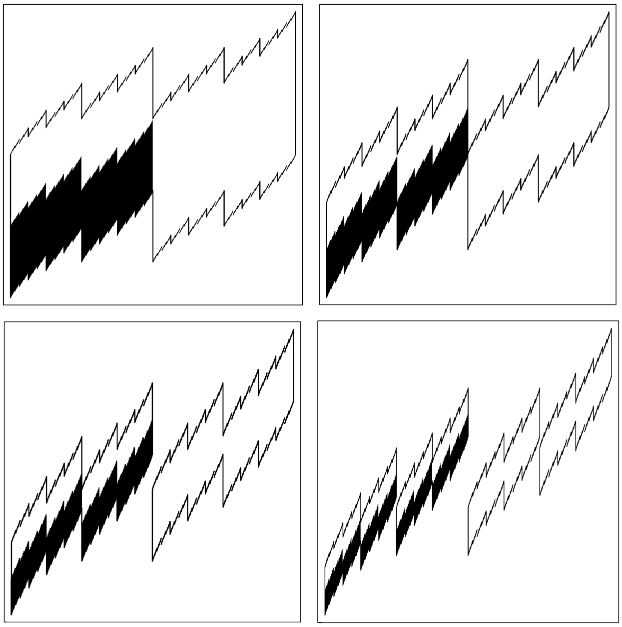

Fig. 1. T and $A^{-1}T$ (the black region) for $a = 1, 2, 3, 5$.

That is, we allow each column of digits move up and down according to the b_i's. The necessary and sufficient condition for the connectedness can be adjusted as follows (likewise for the disk-likeness in the next section).

Theorem 2.1. *Let A be as in* (1.3) *and $\mathcal{D}$ has the form as in* (2.1). *Then T is a connected self-affine tile if and only if*

$$\left|\frac{b_{i+1}-b_i}{q} + \frac{\operatorname{sgn}(p)(b_0 - b_{|p|-1}) - a}{q(q - \operatorname{sgn}(p))}\right| \leqslant 1, \quad 0 \leqslant i < |p| - 1. \tag{2.2}$$

Remark. We claim that it suffices to prove the theorem for $p, q > 0$. Indeed note that $T(A, \mathcal{D}) = T(A^2, A\mathcal{D} + \mathcal{D})$. Here

$$A^2 = \begin{bmatrix} p^2 & 0 \\ -(p+q)a & q^2 \end{bmatrix}$$

and

$$A\mathcal{D} + \mathcal{D} = \left\{ \begin{bmatrix} pr + \ell \\ (-ar + qb_r + b_\ell) + (qj + k) \end{bmatrix} : 0 \leqslant r,\ \ell \leqslant |p| - 1,\ 0 \leqslant j,\ k \leqslant |q| - 1 \right\}.$$

If $p > 0$ and $q < 0$, we let $\widetilde{\mathcal{D}} = A\mathcal{D} + \mathcal{D} + (0, q^2 + q)^t$ and consider $T(A^2, \widetilde{\mathcal{D}})$. It is a translation of $T(A, \mathcal{D})$. By a change of variable on the second coordinate (replace the above j by $(|q| - j)$ and the k by $(|q| - k)$), we have

$$\widetilde{\mathcal{D}} = \left\{ \begin{bmatrix} i \\ b_i' + j \end{bmatrix} : 0 \leqslant i \leqslant p^2 - 1,\ 0 \leqslant j \leqslant q^2 - 1 \right\}$$

where

$$b_i' = -ar + qb_r + b_\ell \quad \text{with } i = pr + \ell,\ 0 \leqslant r,\ \ell < p.$$

Hence $\widetilde{\mathcal{D}}$ has the expression as in (2.1). Furthermore it follows from a direct observation that, for $i = pr + \ell$, $\frac{b'_{i+1} - b'_i}{q^2} + \frac{(b'_0 - b'_{p^2-1}) - (p+q)a}{q^2(q^2-1)}$ is equal to $(\frac{b_{\ell+1} - b_\ell}{q} + \frac{b_0 - b_{p-1} - a}{q(q-1)})\frac{1}{q}$ if $\ell < p - 1$ and to $\frac{b_{r+1} - b_r}{q} + \frac{b_0 - b_{|p|-1} - a}{q(q-1)}$ if $\ell = p - 1$. Hence condition (2.2) in the theorem for $(A^2, \widetilde{\mathcal{D}})$ is equivalent to that for $(A, \mathcal{D})$.

For the case $p < 0$, if $q > 0$, we use $\widetilde{\mathcal{D}} = A\mathcal{D} + \mathcal{D} + (p^2 + p, 0)^t$; if $q < 0$ we use $\widetilde{\mathcal{D}} = A\mathcal{D} + \mathcal{D} + (p^2 + p, q^2 + q)^t$. By applying the same argument as the above, $\widetilde{\mathcal{D}}$ has the expression as in (2.1) with

$$b_i' = a(p + 1 + r) + qb_{-p-1-r} + b_\ell \quad \text{with } i = |p|r + \ell,\ 0 \leqslant r,\ \ell < |p|,$$

and condition (2.2) in the theorem for $(A^2, \widetilde{\mathcal{D}})$ is also equivalent to that for $(A, \mathcal{D})$.

In view of the expression for A^2 above, we can actually assume that $p, q > 0$ in the proof. In this case the $\mathrm{sgn}(p)$ in (2.2) is just 1.

In the sequel, unless otherwise stated we assume without loss of generality that the self-affine pair $(A, \mathcal{D})$ is such that: *A is as in* (1.3) *with $p, q \geqslant 2$, and $\mathcal{D}$ has the form as in* (2.1). We have

$$A^{-1} = \begin{bmatrix} p^{-1} & 0 \\ (pq)^{-1}a & q^{-1} \end{bmatrix} \quad \text{and} \quad A^{-n} = \begin{bmatrix} p^{-n} & 0 \\ r_n a & q^{-n} \end{bmatrix}, \quad n \geqslant 1 \tag{2.3}$$

where

$$r_n = \begin{cases} (p^{-n} - q^{-n})/(q-p), & \text{if } p \neq q, \\ n/p^{n+1}, & \text{if } p = q. \end{cases} \tag{2.4}$$

It is easy to see that $r_n = (p^n q)^{-1} + r_{n-1} q^{-1}$, $n \geqslant 1$ (assume $r_0 = 0$). Let $\mathcal{I}_1$ denote the set of $\mathbf{i} = i_1 i_2 \cdots$ with $i_n \in \mathcal{D}_1 = \{1, \ldots, p-1\}$, and $\mathcal{I}_2$ denote the set of $\mathbf{j} = j_1 j_2 \cdots$ with $j_n \in \mathcal{D}_2 = \{1, \ldots, q-1\}$. Then it follows from (1.2), (2.1) and (2.3) that

$$T = \left\{ \begin{bmatrix} p(\mathbf{i}) \\ ar(\mathbf{i}) + b(\mathbf{i}) \end{bmatrix} + \begin{bmatrix} 0 \\ q(\mathbf{j}) \end{bmatrix} :\ 0 \leqslant i_n < p,\ 0 \leqslant j_n < q \right\} \tag{2.5}$$

where

$$p(\mathbf{i}) = \sum_n \frac{i_n}{p^n}, \qquad r(\mathbf{i}) = \sum_n r_n i_n, \qquad b(\mathbf{i}) = \sum_n \frac{b_{i_n}}{q^n} \quad \text{and} \quad q(\mathbf{j}) = \sum_n \frac{j_n}{q^n}.$$

It follows that the range of the x-coordinate of T is the interval $[0, 1]$. For each $x = p(\mathbf{i})$ such that the representation is unique, the vertical cross section of T is the interval of length 1 with an end point at $ar(\mathbf{i}) + b(\mathbf{i})$; for the other points that have two representations, the vertical cross section of T is the union of two intervals of length 1.

Proposition 2.2. *For the self-affine pair $(A, \mathcal{D})$, T is a tile with Lebesgue measure* 1. *Moreover for any sequence $\{c_n\}_{n\in\mathbb{Z}}$ in $\mathbb{R}$, let $\mathcal{J} = \{(n, c_n + m)^t\colon\ n, m \in \mathbb{Z}\}$, then $\mathcal{J}$ is a tiling set for T in $\mathbb{R}^2$.*

Proof. Let $\mathcal{D}_1 = \{0, 1, \ldots, p-1\}$, $\mathcal{D}_2 = \{0, 1, \ldots, q-1\}$. For any $(x, y)^t \in \mathbb{R}^2$, since $T(p, \mathcal{D}_1) = [0, 1]$, we can find $\ell \in \mathbb{Z}$ such that $x - \ell \in [0, 1]$. Let $\mathbf{i} \in \mathcal{I}_1$ be such that $x - \ell = p(\mathbf{i})$. Let $y_0 = ar(\mathbf{i}) + b(\mathbf{i})$. Since $T(q, \mathcal{D}_2) = [0, 1]$ also, there is $k \in \mathbb{Z}$ such that $y - (y_0 + c_\ell + k) \in T(q, \mathcal{D}_2)$. This implies that $y - (y_0 + c_\ell + k) = q(\mathbf{j})$ for some $\mathbf{j} \in \mathcal{I}_2$. It follows that $(x, y)^t \in T + (\ell, c_\ell + k)^t$ (by (2.5)). Hence $\mathcal{J} + T = \mathbb{R}^2$.

Note that both $T(p, \mathcal{D}_1)$ and $T(q, \mathcal{D}_2)$ are the unit interval $[0, 1]$. We have for almost all $x \in \mathbb{R}$, the above ℓ and $\mathbf{i}$ are unique. If we fix such x, then for almost all $y \in \mathbb{R}$, the above k and y_0 are unique. Therefore, for almost all $(x, y)^t \in \mathbb{R}^2$, the above ℓ and k are unique. Hence the family $\{T + t, t \in \mathcal{J}\}$ are measure disjoint sets. Therefore $T + \mathcal{J}$ tile $\mathbb{R}^2$. That T has Lebesgue measure 1 follows from the fact [13] that $\mathbb{Z}^2$ is a tiling set (taking $c_n = 0$ in the above). $\square$

Geometrically, the tile T has the two sides on the vertical line $x = 0$ and $x = 1$ (by (2.5)). Proposition 2.2 implies that the tiling can be slid vertically. In order to prove the connectedness, we need the following elementary fact (see, for example, [11, Theorem 4.3]):

Lemma 2.3. *Let $\{\psi_j(x)\}_{j=1}^N$ be a family of contractions on $\mathbb{R}^n$ and let K be its attractor. Then K is connected if and only if, for any $i \neq j \in \{1, 2, \ldots, N\}$, there exists a sequence $i = j_1, j_2, \ldots, j_n = j$ of indices in $\{1, 2, \ldots, N\}$ so that $\psi_{j_k}(K) \cap \psi_{j_{k+1}}(K) \neq \emptyset$ for all $1 \leqslant k < n$.*

For the affine pair $(A, \mathcal{D})$, we let

$$S_{i,j}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} i \\ b_i + j \end{bmatrix}\right),$$

where $0 \leqslant i < p$, $0 \leqslant j < |q|$. Then $\{S_{i,j}\}_{i,j}$ is the iterated function system (IFS) that generates T. In view of (2.3) and (2.5), the elements of $S_{i,j}(T)$ are of the form

$$\begin{bmatrix} p(i\mathbf{i}) \\ ar(i\mathbf{i}) + b(i\mathbf{i}) \end{bmatrix} + \begin{bmatrix} 0 \\ q(j\mathbf{j}) \end{bmatrix} \tag{2.6}$$

where $\mathbf{i} = i_1 i_2 \cdots \in \mathcal{I}_1$ and $\mathbf{j} = j_1 j_2 \cdots \ \in \mathcal{I}_2$.

Lemma 2.4. *Let $(A, \mathcal{D})$ be the self-affine pair as above. Then $S_{i,j}(T) \cap S_{i,j+1}(T)$ contains an interior point of T.*

Proof. Let e_i, $i = 1, 2$ be the two unit vectors of $\mathbb{R}^2$. To prove the lemma, it suffices to show that $T \cap (T + e_2)$ contains a point of $(T \cup (T + e_2))^o$.

Observe that for any $\mathbf{i} \in \mathcal{I}_1$ and $\mathbf{j} \in \mathcal{I}_2$, $p(\mathbf{i}), q(\mathbf{j}) \in [0, 1]$ (as $p, q > 0$). We fix an $0 < x_0 < 1$ so that there is a unique $\mathbf{i} \in \mathcal{I}_1$ such that $x_0 = p(\mathbf{i})$. Let

$$y_0 = ar(\mathbf{i}) + b(\mathbf{i}) + 1. \tag{2.7}$$

In view of (2.5), we have $(x_0, y_0)^t \in T \cap (T + e_2)$.

Next we show that $(x_0, y_0)^t$ is in the interior of $(T \cup (T + e_2))$. By Proposition 2.2, $\mathbb{Z}^2$ is a tiling set of T. Since x_0 is an interior point of $[0, 1]$, by (2.5), $(x_0, y_0)^t \in T + (n, m)^t$ only if $n = 0$. For $(x_0, y_0)^t \in T + (0, m)^t$, y_0 must be the form

$$y_0 = ar(\mathbf{i}) + b(\mathbf{i}) + q(\mathbf{j}) + m$$

for some $\mathbf{j}$ (note that $\mathbf{i}$ is uniquely determined). Combining (2.7) and (2.5), we have $q(\mathbf{j}) + m = 1$ and m must equal 0 or 1 (as $0 \leqslant q(\mathbf{j}) \leqslant 1$). It follows that $(x_0, y_0) \notin T + me^2$ for any $m \notin \{0, 1\}$, therefore (x_0, y_0) must be in $(T \cup (T + e_2))^o$. □

Lemma 2.5. *With the self-affine pair $(A, \mathcal{D})$ as the above, let $G_i = \bigcup_{j=0}^{q-1} S_{i,j}(T)$, $i = 0, 1, \ldots, p-1$. Then $G_i(T) \cap G_\ell(T) \neq \emptyset$ implies $|i - \ell| \leqslant 1$.*

Proof. By (2.6), G_i has the following form

$$G_i = \left\{ \begin{bmatrix} p(i\mathbf{i}) \\ ar(i\mathbf{i}) + b(i\mathbf{i}) \end{bmatrix} + \begin{bmatrix} 0 \\ q(\mathbf{j}) \end{bmatrix} : \mathbf{i} \in \mathcal{I}_1, \mathbf{j} \in \mathcal{I}_2 \right\}. \tag{2.8}$$

From the expression of the first coordinate, G_i is the part of T between the vertical lines $x = i/p$ and $x = (i+1)/p$. The lemma follows. □

Lemma 2.6. *For the above G_i, $i = 0, \ldots, p-2$, we have $G_i(T) \cap G_{i+1}(T)$ is a line segment if and only if*

$$\left| \frac{b_{i+1} - b_i}{q} + \frac{(b_0 - b_{p-1}) - a}{q(q-1)} \right| < 1,$$

and is a single point if equality holds.

Proof. From the proof of Lemma 2.5, we see that the intersection $S_{i,j}(T) \cap S_{i+1,k}(T)$ has a unique first coordinate $(i+1)/p$. By (2.8), all digits in $\mathbf{i}$ are $p-1$. This implies that the second coordinate of an element of G_i is of the form (use (2.8))

$$y_i = a\left(\frac{i}{pq} + (p-1)\sum_{n=2}^{\infty} r_n \right) + \left(\frac{b_i}{q} + \sum_{n=2}^{+\infty} \frac{b_{p-1}}{q^n} \right) + q(\mathbf{j})$$

where $\mathbf{j} = j_1 j_2 \cdots \in \mathcal{I}_2$. By using (2.4), it is direct to show that

$$\sum_{n=2}^{\infty} r_n = \frac{p+q-1}{pq(p-1)(q-1)}.$$

Also note that $\{q(\mathbf{j}) \colon \mathbf{j} \in \mathcal{I}_2\} = [0, 1]$, the above y_i's form a unit interval

$$I_1 = [\alpha, \alpha + 1] := \left(\frac{(i+1)a}{pq} + \frac{b_i}{q} \right) + \frac{b_{p-1} + a}{q(q-1)} + [0, 1].$$

Similarly, the second coordinate of elements in G_{i+1} (the first coordinate is $(i+1)/p$) has the form

$$y_{i+1} = \left(\frac{(i+1)a}{pq} + \frac{b_{i+1}}{q} \right) + \frac{b_0}{q(q-1)} + q(\mathbf{j}), \tag{2.9}$$

which determines the unit interval

$$I_2 = [\beta, \beta + 1] := \left(\frac{(i+1)a}{pq} + \frac{b_{i+1}}{q} \right) + \frac{b_0}{q(q-1)} + [0, 1].$$

It follows that if $G_i \cap G_{i+1} \neq \emptyset$, then the first coordinate of the intersection is $(i+1)/p$, and the second coordinate is $I_1 \cap I_2$. Note that $[\alpha, \alpha + 1] \cap [\beta, \beta + 1]$ is an empty set when $|\alpha - \beta| > 1$; a single point when $|\alpha - \beta| = 1$ and an interval when $|\alpha - \beta| < 1$. The proposition follows by observing that $\alpha - \beta = \frac{b_i - b_{i+1}}{q} + \frac{b_{p-1} - b_0 + a}{q(q-1)}$. □

Proof of Theorem 2.1. Let $T=\bigcup_i G_i$ be as in Lemma 2.5. Assume that T is connected, then for any i, G_i must intersect some G_ℓ, $\ell \neq i$. The necessity follows from Lemmas 2.5 and 2.6.

To prove the sufficiency, we have from Lemma 2.4,

$$S_{i,j}(T)\cap S_{i,j+1}(T)\neq\emptyset, \quad 0\leqslant j<q-1. \tag{2.10}$$

On the other hand, by Lemma 2.6 and the definition of G_i, we have for each $0\leqslant i<p-1$, there exist $0\leqslant j_i, k_i<q$ such that

$$S_{i,j_i}(T)\cap S_{i+1,k_i}(T)\neq\emptyset. \tag{2.11}$$

We use (2.10) and (2.11) to select a sequence $\{\psi_i\}_{i=1}^N$ from $\{S_{ij}\}_{i,j}$ in the following order (a zigzag path): $S_{0,0}, S_{0,1},\dots, S_{0,|q|-1}, S_{0,|q|-2},\dots,S_{0,j_0}, S_{1,k_0},\dots,S_{1,|q|-1}, S_{1,|q|-2},\dots,S_{1,0}, S_{1,1},\dots,S_{1,j_1}, S_{2,k_1},\dots,S_{2,|q|-1}, S_{2,|q|-2}, S_{2,|q|-2},\dots,S_{2,j_2}, \dots, S_{p-1,|q|-1}$, where, j_i and k_i are given by (2.11). Then each $S_{i,j}$ appears at least once in the sequence $\{\psi_i\}_{i=1}^N$ and

$$\psi_i(T)\cap\psi_{i+1}(T)\neq\emptyset \quad \forall 1\leqslant j<N.$$

This implies that T is connected by using Lemma 2.3, and the sufficiency is proven. □

3. The disk-likeness

We prove the following theorem on the disk-likeness of T:

Theorem 3.1. *Let $(A,\mathcal{D})$ be as in Theorem 2.1. Then the self-affine tile T is disk-like if and only if*

$$\left|\frac{b_{i+1}-b_i}{q}+\frac{\operatorname{sgn}(p)(b_0-b_{|p|-1})-a}{q(q-\operatorname{sgn}(p))}\right|<1 \tag{3.1}$$

for all $i=0,1,\dots,|p|-2$.

We adopt the same notations and assume $p,q\geqslant 2$ as in the last section. We need a lemma for the proof of the sufficiency.

Lemma 3.2. *Suppose inequality (3.1) holds, then for each $0\leqslant i<p-1$, there are j_i, k_i such that $S_{i,j_i}(T)\cap S_{i+1,k_i}(T)$ contains an interior point of $S_{i,j_i}(T)\cup S_{i+1,k_i}(T)$.*

Proof. In view of Lemma 2.6 and the assumption, there exist j_i and k_i such that the intersection $S_{i,j_i}(T)\cap S_{i+1,k_i}(T)$ contains a vertical non-degenerated line segment. Let $\hat{z}=S_{i,j_i}(\tilde{z})$ be the mid-point of the line segment. By the definition of $S_{i,j}$, we see that

$$S_{i,j_i}(T)=A^{-1}\left(T+\begin{bmatrix} i\\ b_i+j_i\end{bmatrix}\right),$$
$$S_{i+1,k_i}(T)=A^{-1}\left(T+\begin{bmatrix} i\\ b_i+j_i\end{bmatrix}+\begin{bmatrix} 1\\ c\end{bmatrix}\right),$$

where $c=b_{i+1}-b_i+k_i-j_i$. Hence $T\cap(T+(1,c)^t)$ is a vertical line segment with positive length and $\tilde{z}$ is its mid-point. Note that the left side and the right side of T are vertical line segments with length 1, it is easy to show that $\tilde{z}\in T+(\ell,\ c+k)$ if and only if $(\ell,c+k)$ equals $(0,0)$ or $(1,c)$. By Proposition 2.2, we see that $\tilde{z}$ is an interior point of $T\cup(T+(1,c))^t$. This means that $\hat{z}$ is a point of $(S_{i,j_i}(T)\cup S_{i+1,k_i}(T))^o$. □

Corollary 3.3. *Under the assumption (3.1), T^o is connected.*

Proof. Let ψ_i be the maps defined as in the proof of Theorem 2.1 (where j_i and k_i are chosen as in Lemma 3.2). Then $\psi_i(T)\cap\psi_{i+1}(T)$ contains at least one interior point of T. As $T=\bigcup_i\psi_i(T)$, we conclude that T^o is connected by Lemma 2.3. □

Proof of Theorem 3.1. Since T is a self-affine tile (Proposition 2.2), so the iterated function system $\{S_{ij}\colon 0\leqslant i<p,\ 0\leqslant j<q\}$ satisfies the open set condition. Assuming condition (3.1), then Corollary 3.3 implies that T^o is connected, which yields the disk-likeness of T by a theorem of Luo et al. [17].

For the necessity suppose T is disk-like, then both T and T^o are connected and Theorem 2.1 ensures that $|\frac{b_{i+1}-b_i}{qt}+\frac{(b_0-b_{p-1})-a}{q(q-1)}|\leqslant 1$ for $i=0,\dots,p-2$. If the equality holds for some i, then the second part of Lemma 2.6 implies that

$G_i \cap G_{i+1}$ contains only one point. On the other hand, Lemma 2.5 implies that $G_i \cap G_j = \emptyset$ for $|i-j| \geqslant 2$. Since $T = \bigcup_{j=0}^{p-1} G_j$, T can be divided into two parts $\bigcup_{j=0}^{i} G_i$ and $\bigcup_{j=i+1}^{p-1} G_i$ with only one common point, this point must be at the boundary of T. Therefore T^o is not connected. This contradicts the connectedness of T^o, and the necessity follows. □

4. The disconnected case

In this section, we assume the self-affine pair $(A, \mathcal{D})$ is as in (1.3) and prove Theorem 1.2. By the same reason as before, we need only prove the case where $p, q \geqslant 2$. Therefore the condition on A is reduced to $q^{m-1} < |\frac{a}{q(q-1)}| \leqslant q^m$.

Proof of Theorem 1.2. Similar to the G_i in Lemma 2.4, we define, for any finite sequence $\mathbf{i} = i_1 i_2 \cdots i_m$, $0 \leqslant i_n < p$,

$$G_{\mathbf{i}} = \bigcup_{j_1=0}^{q-1} \cdots \bigcup_{j_m=0}^{q-1} S_{i_1,j_1} \circ S_{i_2,j_2} \circ \cdots \circ S_{i_m,j_m}(T).$$

Then

$$T = \bigcup_{0 \leqslant i_j < p} G_{i_1 i_2 \cdots i_m}. \tag{4.1}$$

Note that each point in $G_{\mathbf{i}}$ has first coordinate in the interval $[p(\mathbf{i}), p(\mathbf{i}) + 1/p^m]$ where $p(\mathbf{i}) = \sum_{n=1}^{m} i_n/p^n$; $G_{\mathbf{i}}$ is the part of T between the vertical lines $x = p(\mathbf{i})$ and $x = p(\mathbf{i}) + 1/p^m$. By (1.2), it is direct to show that $G_{\mathbf{i}}$ has the following form

$$G_{\mathbf{i}} = \left\{ \sum_{n=1}^{m} A^{-n} \begin{bmatrix} i_n \\ v_n \end{bmatrix} + \sum_{n=m+1}^{\infty} A^{-n} \begin{bmatrix} u_n \\ v_n \end{bmatrix} \colon 0 \leqslant u_n < p,\ 0 \leqslant v_n < q \right\}. \tag{4.2}$$

Note that $\sum_{n=1}^{\infty} A^{-n}(0, v_n)^t = \sum_{n=m+1}^{\infty} A^{-n}(0, q^m v_n)^t$. We can rewrite

$$G_{\mathbf{i}} = \begin{bmatrix} p(\mathbf{i}) \\ 0 \end{bmatrix} + A^{-m} P\big(T\big(P^{-1}AP, \mathcal{D}\big)\big) \quad \text{with } P = \begin{bmatrix} 1 & 0 \\ 0 & q^m \end{bmatrix}.$$

By applying Theorem 1.1 to $T(P^{-1}AP, \mathcal{D})$, we see that $G_{\mathbf{i}}$ is connected if and only if $|\frac{aq^{-m}}{q(q-1)}| \leqslant 1$. This implies that $G_{\mathbf{i}}$ is connected.

Next we replace the above m by $k \leqslant m$, then from the assumption, $1 < |aq^{-k}/(q-1)|$. We show that if $0 \leqslant \ell \leqslant p-1$, $\ell \neq i_k$, then $G_{i_1 \cdots i_k} \cap G_{i_1 \cdots i_{k-1} \ell} = \emptyset$. For k runs through 1 to m, we see that $G_{i_1 \cdots i_m}$ are connected components and there is a total p^m of them, as stated in the theorem.

Suppose otherwise there exist $k \leqslant m$ and $i_k < \ell$ such that $G_{i_1 \cdots i_k} \cap G_{i_1 \cdots i_{k-1} \ell} = \emptyset$. Then (2.3) and (4.2) imply that there exist $0 \leqslant u_n$, $s_n < p$ and $0 \leqslant v_n$, $t_n < q$ such that (the first coordinate)

$$p^{-k} i_k + \sum_{n=k+1}^{\infty} p^{-n} u_n = p^{-k} \ell + \sum_{n=k+1}^{\infty} p^{-n} s_n, \tag{4.3}$$

and (the second coordinate)

$$\sum_{n=1}^{k} a r_n i_n + \sum_{n=k+1}^{\infty} a r_n u_n + \sum_{n=1}^{\infty} q^{-n} v_n = \sum_{n=1}^{k-1} a r_n i_n + a r_k \ell + \sum_{n=k+1}^{\infty} a r_n s_n + \sum_{n=1}^{\infty} q^{-n} t_n. \tag{4.4}$$

Then (4.3) implies that $\ell = i_k + 1$ and $u_n = p-1$, $s_n = 0$ for all $n > k$. Hence (4.4) becomes

$$\sum_{n=k+1}^{\infty} a r_n (p-1) + \sum_{n=1}^{\infty} q^{-n} v_n = a r_k + \sum_{n=1}^{\infty} q^{-n} t_n.$$

Since both $\sum_{n=1}^{\infty} q^{-n} t_n$ and $\sum_{n=1}^{\infty} q^{-n} v_n$ belong to $[0, 1]$, we have

$$\left| \sum_{n=k+1}^{\infty} a r_n (p-1) - a r_k \right| \leqslant 1.$$

On the other hand by (2.4) and a direct calculation,

$$\sum_{n=k+1}^{\infty} ar_n(p-1) - ar_k = \frac{aq^{-k}}{q-1}.$$

Hence $|\frac{aq^{-k}}{q-1}| \leqslant 1$, a contradiction. Therefore, $G_{i_1\cdots i_k} \cap G_{i_1\cdots i_{k-1}\ell} = \emptyset$ as claimed. $\square$

The above proof and Theorem 1.1 (the disk-likeness part) also yield the following conclusion.

Corollary 4.1. *Let $(A, \mathcal{D})$ be as in Theorem 1.1. Suppose $|q|^{m-1} < |\frac{a}{q(q-\operatorname{sgn}(p))}| < |q|^m$ for some $m > 0$, then T is a tile and consists of p^m disjoint disk-like components.*

5. Remarks

We remark that Theorem 1.2 can also be proved for the more general digit set in (2.1), the proof is similar but the expression of the condition in the theorem is more complicated.

In regard to the example in the introduction with $A = \begin{bmatrix} 2 & 0 \\ -a & 2 \end{bmatrix}$, the b_0, b_1 do not affect the connectedness and disk-likeness of T, as the conditions in (2.2) and (3.1) are reduced to $|a/2| \leqslant 1$ and $|a/2| < 1$ respectively. Moreover, if $p = q$ $(\geqslant 2)$ and $b_i = c + bi$, $i = 0, \ldots, p-1$, for some constants c and b, then these b_i do not affect the connectedness and disk-likeness of T. This observation does not hold if $p \neq q$.

Let A be as in (1.3), we consider $\mathcal{D} = \{(d_i, e_j)^t\colon\ 1 \leqslant i \leqslant \ell,\ 1 \leqslant j \leqslant k\}$ with $k\ell = |pq|$ as any other product form digit set. If $\ell < |p|$, then the set $\{x \in \mathbb{R}\colon\ (x, y)^t \in T \text{ for some } y \in \mathbb{R}\}$ has one-dimensional Lebesgue measure zero, so T cannot be a tile. For the same reason if $a = 0$, then T cannot be a tile when $k < |q|$. On the other hand if $a \neq 0$ and $k < |q|$, then T can still be a tile. For example, let $a = 1$ in the lower triangular matrix A, let $\mathcal{D} = \{(i, 0)\colon\ 0 \leqslant i < |pq|\}$ (i.e., $k = 1$ and $\mathcal{D}$ is a consecutive collinear digit set), then T is a connected self-affine tile [11]. For the lower triangular matrix A, we do not know a general condition on $\mathcal{D}$ to ensure that T is a tile or a connected tile.

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